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Semi-classical mechanics in phase space: the quantum target of minimal strings

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Abstract: The target space $M_{p,q}$ of (p,q) minimal strings is embedded into the phase space of an associated integrable classical mechanical model. This map is derived from the matrix model representation of minimal strings. Quantum effects on the target space are obtained from the semiclassical mechanics in phase space as described by the Wigner function. In the classical limit the target space is a fold catastrophe of the Wigner function that is smoothed out by quantum effects. Double scaling limit is obtained by resolving the singularity of the Wigner function. The quantization rules for backgrounds with ZZ branes are also derived.

Keywords: Bosonic Strings, Matrix Models.

Contents

1. Introduction

The recent progress in the study of Liouville theory [1] opens the possibility to address some problems of quantum gravity using as a theoretical laboratory minimal (p, q) strings [2, 3]. The information about the minimal string target space is encoded in the dynamics of the FZZT branes [4]. In this paper we present a new approach to study quantum corrections to the minimal string target geometry for $(2, q)$ models.

The strategy we follow is to map the minimal string data about FZZT brane amplitudes into an integrable classical mechanical model. By this procedure we map the minimal string target into a curve in the mechanical model phase space. The form of this map can be directly derived from the matrix model representation of the minimal string. Once we have embedded the classical string target into the phase space of the auxiliary mechanical model, quantum effects on the target, i.e quantum gravity effects, are derived from the semi-classical mechanics on phase space as it is described by the Wigner function of the mechanical system.[1] In the classical $h = 0$ limit the support of the Wigner function is concentrated on the classical target submanifold. Moreover the classical target defines a catastrophe singularity for the Wigner function that is smoothed out by quantum effects [8]. The semi-classical Wigner function spreads out on phase space in the way dictated by the type of singularity associated with the classical target. In particular, the exact double scaling limit is obtained by resolving the fold singularity of the Wigner function at the classical turning point. This leads to an interesting connection between catastrophe theory and double scaling limit. The Stokes's phenomenon described in [4] enters into this picture in a very natural way dictating the type of saddles contributing to the Wigner function in the classically forbidden region of phase space.

In this approach ZZ branes are related to the type of singularity at the turning point. For backgrounds with ZZ branes a quantization rule involving the number of ZZ branes emerges naturally. This leads to fix the relation between the Planck constant of the auxiliary mechanical model and the string coupling constant.

The plan of the paper is the following. In section 3 we introduce the auxiliary mechanical model. In section 4 we define the Wigner function and describe the type of singularity of this function on target space. In section 5 we derive the map from minimal strings into the auxiliary classical mechanical model from the matrix model description of minimal strings. In section 6 we derive the double scaling limit solution by an uniform approximation to the Wigner function near the singularity. We work out the gaussian model in full detail. In section 7 we consider the issue of ZZ brane backgrounds, we derive the quantization rules and relate the type of singularity at the classical turning point with the types of different ZZ branes.

2. The classical target of minimal strings

For minimal strings defined by coupling (p, q) minimal models to Liouville theory, a Riemann surface $M_{p,q}$, playing the role of a classical target space time, can be naturally defined using the FZZT amplitude on the disk [2]. Let $\Phi(\mu, \mu_b)$ be the FZZT disk amplitude for μ and μ_b the bulk and boundary cosmological constants. Defining

$$x = \frac{\mu_b}{\sqrt{\mu}} \tag{2.1}$$

and

$$y = \frac{1}{(\sqrt{\mu})^{1+\frac{1}{b^2}}} \partial_x \Phi \tag{2.2}$$

the Riemann surface $M_{p,q}$ is given by

$$F(x, y) = T_q(x) - T_p(y) = 0\,, \tag{2.3}$$

where T_p are the Chebyshev polynomials of the first kind. Singular points of $M_{p,q}$, defined by $F(x, y) = \partial_x F = \partial_y F = 0$, are one to one related to eigenstates of the ground ring

[1] The information encoded in the Wigner function is, in this interpretation, the same type of information you will expect to derive from a Hartle-Hawking wave function [5].

(principal ZZ brane states). In this geometrical interpretation we have

$$\Phi(x) = (\sqrt{\mu})^{1+\frac{1}{b^2}} \int_P^x y dx \tag{2.4}$$

and for the ZZ amplitude

$$\Phi_{ZZ} = (\sqrt{\mu})^{1+\frac{1}{b^2}} \oint y dx \,, \tag{2.5}$$

where the integration closed path passes through the singular point. As it is clear from (2.4), the curve $M_{p,q}$ can be interpreted as the moduli of FZZT branes [4] and therefore as a model for the target space of the corresponding minimal string.

3. The auxiliary classical mechanical system

Let us consider the simplest integrable system with just one constant of motion, the energy E, corresponding to a non relativistic particle in a potential $V(q)$. Let us denote $p(q,E)$ the two valued function giving the momentum. We define the reduced action $S(q)$ as

$$S(q,E) = \int_P^q p(q',E)dq' \,. \tag{3.1}$$

The one dimensional "torus" in phase space, which we will denote Σ_E, is defined by the curve

$$p = p(q,E) \,. \tag{3.2}$$

The motion in phase space is confined to lay on this torus. For a bounded system the action variable is defined by

$$I = \frac{1}{2\pi} \oint p(q)dq \,, \tag{3.3}$$

where the integral is on the homology cycles of the "torus", i.e. Σ_E itself for the one dimensional case, and the angle variable by

$$\theta = \partial_I S(q,I) \,. \tag{3.4}$$

For an unbounded system we define I to be simply the energy E and θ the uniformizing parameter of the open "torus".

In what follows we will associate with the $(2, q = 2k-1)$ minimal string data $\Phi(x)$ and the curve $M_{p,q}$ a classical one dimensional unbounded integrable model by the following map

$$\Phi(x) = iS(q, E = 0) \tag{3.5}$$

with $q = x$ and S the analytic continuation of (3.1). By this map the curve $M_{p,q}$, defined by $y = y(x)$, is transformed into the curve $p = p(q, E = 0)$ in phase space, i.e the torus associated with the integral of motion E.

As a concrete example, we can consider the curve $M_{2,1}$. It is given by

$$y = \pm\sqrt{\frac{x+1}{2}} \tag{3.6}$$

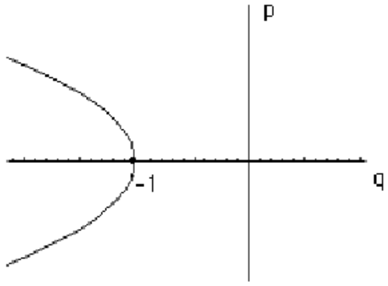

Figure 1: Phase space curve $p(q, E = 0)$ for (2,1) minimal string.

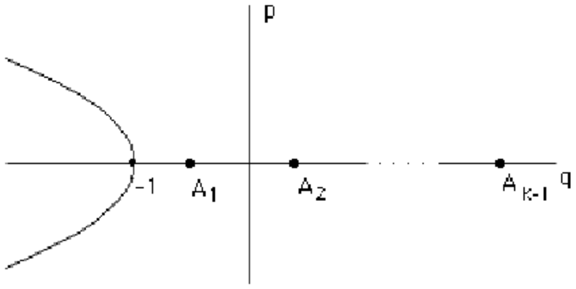

Figure 2: Phase space curve $p(q, E = 0)$ for (2,2k-1) minimal string.

which leads, by using (3.5), to a classical mechanical model defined by

$$[V(q) - E]\,|_{E=0} = -\frac{x}{2} - \frac{1}{2} \tag{3.7}$$

with the curve $p(q, E = 0)$ as depicted in figure 1. In general, for the $(2, 2k - 1)$ minimal string we have

$$y = \pm\sqrt{2^{2k-3}(x+1)\prod_{n=1}^{k-1}(x - x_{1,n})^2}\,, \tag{3.8}$$

where $x_{1,l} = -\cos\frac{2\pi l}{2k-1}$. The curve $p(q, E = 0)$ on phase space has the structure depicted in figure 2. (3.8) has singular points at $y = 0$, $x = x_{1,l}$, denoting the existence of ZZ brane states. The existence of these singular points is reflected in the $p(q, E = 0)$ curve as the extra points A_l in figure 2.

4. Quantization of minimal strings

As a first step towards the quantization of minimal strings we will consider the semi-classical quantization of the auxiliary mechanical model. This approach leads to map higher genus corrections to the minimal string theory into quantum corrections of the auxiliary mechanical system. More precisely we will identify the wave function of the eigenstate $E = 0$ with the all genus FZZT partition function.

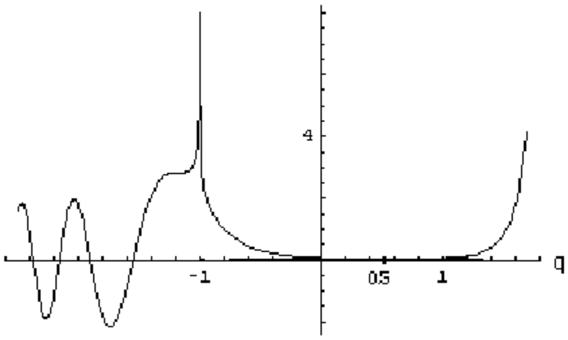

Figure 3: $\psi_{E=0}$ in WKB approximation for the (2,3) system.

In order to quantize the analog mechanical model we first define the associated Heisenberg algebra

$$[\hat{q},\hat{p}] = i\hbar\,. \tag{4.1}$$

The meaning of this Heisenberg algebra as well as the Planck constant will become clear in the context of matrix models in next section. For the given value $E = 0$ we define in WKB

$$Z_{FZZT}(x) \equiv \psi_{E=0}(q) \approx \left| \frac{\partial^2 S}{\partial_q \partial_E}(q, E=0) \right|^{1/2} e^{\frac{i}{\hbar}S(q,E=0)}\,. \tag{4.2}$$

In the simplest $(2,1)$ model we can easily get the exact eigenstate. It is given by

$$\psi_{E=0}(q) = Ai\left(\frac{q+1}{2^{1/3}\hbar^{2/3}}\right) \tag{4.3}$$

the Airy function. For the $(2,3)$ model we get in WKB the function depicted in figure 3. Notice that this function diverges at $q = \infty$. Due to reasons we will explain below, we take the asymptotic behavior ansatz in such a way that, for the $(2,2k-1)$ model, $\psi_{E=0}(q)$ vanishes at $q = +\infty$ for k odd. For k even, which corresponds to nonperturbatively inconsistent models, $\psi_{E=0}(q)$ diverges at $q = +\infty$.

4.1 The quantum corrected target and the fold catastrophe

In order to see the quantum fate of the curve $M_{p,q}$, defined by $y = y(x)$ or by the mechanical analog torus $p = p(q,E)$, we use the quantum corrections to the Wigner distribution function $f(p,q)$ [8]. As it is well known, the Wigner function is the quantum mechanical generalization of the classical Boltzmann distribution function on phase space. It is defined by

$$f_E(p,q) = \frac{1}{\pi\hbar}\int dX e^{-\frac{2i}{\hbar}pX}\psi_E(q+X)\psi_E^*(q-X)\,. \tag{4.4}$$

The reason we are interested in the Wigner function is because it is a very natural way to study quantum deformations of the curve $p(q,E)$ on phase space. In fact in the classical limit it is easy to see that $f(p,q)$ is a delta function on the curve $p(q,E)$.

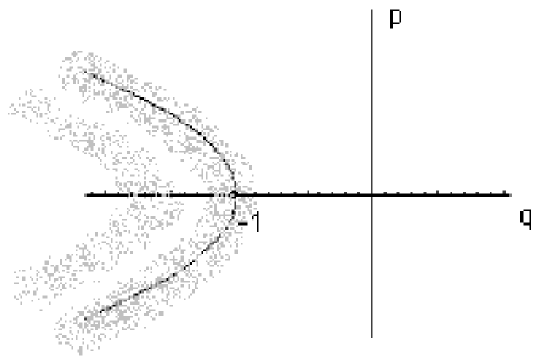

Figure 4: The curve in phase space gets fuzzy by quantum corrections.

Using the WKB approximation for $\psi_E(q)$ we get for the Wigner function the following integral representation

$$f_E(p,q) = \frac{1}{\pi\hbar}\int_{-\infty}^{+\infty} dX \frac{\exp\left[\frac{i}{\hbar}\int_{q-X}^{q+X} p(q',E)dq' - 2pX\right]}{\left|\frac{\partial I}{\partial p}(q+X,p(q+X))\,\frac{\partial I}{\partial p}(q-X,p(q-X))\right|^{1/2}}. \tag{4.5}$$

Given a generic point (p,q), the saddle point approximation to (4.5) is determined by the two solutions X_1 and X_2 of

$$p(q+X,E) + p(q-X,E) = 2p\,. \tag{4.6}$$

It is easy to see that these saddle points coalesce when (p,q) is on the curve $p = p(q,E)$. The net effect of this is that the curve $p = p(q,E)$ defines a fold catastrophe for the Wigner function. The resolution of this catastrophe is given in terms of the Airy function

$$f(p,q) = \frac{\sqrt{2}\left[\frac{3}{2}A(q,p)\right]^{1/6} Ai\left(-\left[\frac{3A(q,p)}{2\hbar}\right]^{2/3}\right)}{\pi\hbar^{2/3}\left|\frac{\partial I}{\partial q}(2)\frac{\partial I}{\partial p}(1) - \frac{\partial I}{\partial q}(1)\frac{\partial I}{\partial p}(2)\right|^{1/2}} \tag{4.7}$$

where $A(q,p) = \int_{q-X}^{q+X} p(q',E)dq' - 2pX$. The net result for a bounded system, with $p = p(q,E)$ a closed curve, is an oscillatory behaviour for the points inside the curve and exponentially decreasing behaviour for the points in phase space outside the curve. It is important to keep in mind that the Airy function representing the quantum corrections to the Wigner distribution function near the curve $p = p(q,E)$ is a consequence of the fold catastrophe on this curve and is generic for any one dimensional integrable mechanical model.

When we translate the curve $p = p(q,E)$ into the associated minimal string curve $y = y(x)$, what we observe is that quantum effects induce a pattern of fringes on the phase space. From this point of view, where the classical target space of the minimal string, defined by the curve $y = y(x)$, is interpreted as the support of the Wigner function in the classical $h = 0$ limit, the effect of quantum corrections on the target space is not to

JHEP11(2005)049

change this curve into another curve but to spread the curve on the phase space in the way dictated by the type of catastrophe, a fold for one dimensional models, defined by the classical curve $y = y(x)$ (see figure 4).

4.2 Quantum target and Stokes' phenomenon

As we described in the previous section, the resolution of the fold catastrophe on the curve $p = p(q, E)$ leads to a Wigner function defined in terms of the Airy function. The role of the Stokes' phenomenon [4] in this context is transparent from a geometrical point of view. From the integral representation (4.5) of the Wigner function we observe that, for points in the convex complement of the concave set defined by the curve $p = p(q, E)$, the solutions to the saddle point equation (4.6) are complex, leading to an imaginary contribution to the exponent in (4.5). The Stokes' phenomenon implicit in the Airy function provides the apropiated asymptotic form on the convex side, namely exponential decay.

For $(2, 2k-1)$ models we have seen in section 3 that the curve $p = p(q, E)$ contains in addition a set of $k-1$ discrete points on the convex side related to the existence of different types of ZZ branes. In this case the Wigner function, as well as the wave function $\psi_E(q)$, has oscillatory behavior in the concave side. Since we can always consider a deformation of the model that sends the discrete points to infinity, obtaining the $(2, 1)$ model, we have exponential decay on the convex side infinitesimaly close to the curve. This implies that the asymptotic form on the convex side at infinity decay exponentially only if k is odd.[2]

5. Matrix models and the mechanical analog

A non perturbative definition of (p, q) minimal string theory can be given by the double scaling limit (d.s.) of certain matrix models[3]. In this section we consider one matrix models corresponding to $(2, q = 2k-1)$ minimal strings

$$Z_m = \frac{1}{\mathrm{vol}(\mathrm{U}(N))} \int dM e^{-\frac{1}{g_m} \operatorname{tr} V(M)} . \tag{5.1}$$

The goal of this section is to derive, from the matrix model point of view, the map defined in (3.5).

5.1 Wigner distribution formalism

Let us denote $\Pi_n(\lambda) = \frac{\lambda^n}{\sqrt{h_n}} + \cdots$ the orthonormal polynomials of the one matrix model and let us introduce the complete set of states $|\psi_n\rangle$ as

$$\langle \lambda | \psi_n \rangle = \psi_n(\lambda) = \Pi_n(\lambda) e^{-\frac{V(\lambda)}{2g_m}} . \tag{5.2}$$

If we define the operator $\hat{q}$ by

$$\langle \psi_m | \hat{q} | \psi_n \rangle = \sqrt{\frac{h_m}{h_{m-1}}} \delta_{m,n+1} + \sqrt{\frac{h_n}{h_{n-1}}} \delta_{m+1,n} \tag{5.3}$$

[2] The WKB expression for the wave function is $\psi_{E=0} \sim (x+1)^{-1/4} \exp\left[-\hbar^{-1} \int y dq\right]$ with the branch of y in the integral changing in every singular point.

[3] See [6, 7] and references therein

JHEP11(2005)049

the recurrence relation of the orthonormal polynomials can be expressed as

$$\langle\lambda|\hat{q}|\psi_n\rangle = \lambda\psi_n(\lambda) \tag{5.4}$$

that is, $\hat{q}$ acts like the position operator. The corresponding momentum operator $\hat{p}$ can be defined by

$$\langle\lambda|\hat{p}|\psi_n\rangle = -ig_m\frac{\partial}{\partial\lambda}\psi_n(\lambda)\,. \tag{5.5}$$

One can consider $|\psi_n\rangle$ as defining the Hilbert space of a one particle quantum mechanical system with the coupling constant g_m playing the role of the Planck constant $\hbar$. Now, if we define a mixed state whose density operator is

$$\hat{\rho} = \sum_{n=0}^{N-1}\frac{1}{N}|\psi_n\rangle\langle\psi_n| \tag{5.6}$$

we easily get

$$\left\langle\frac{1}{N}\,\mathrm{tr}\,M^k\right\rangle = \mathrm{tr}(\hat{\rho}\hat{q}^k)\,. \tag{5.7}$$

The corresponding Wigner function of this mixed state is

$$f(p,q) = \frac{1}{\pi\hbar}\int dXe^{-\frac{ipX}{\hbar}}\langle q+X|\hat{\rho}|q-X\rangle \tag{5.8}$$

from which we obtain that every matrix model correlator can be obtained from $f(p,q)$ as a statistical mean value over the ensemble (5.6)

$$\left\langle\frac{1}{N}\,\mathrm{tr}\,M^k\right\rangle = \int dpdqf(p,q)q^k\,. \tag{5.9}$$

In particular for the resolvent we have

$$R(x) = \int d\lambda\frac{\rho_\hbar(\lambda)}{x-\lambda} = \int dpdqf(p,q)\frac{1}{x-q} \tag{5.10}$$

which leads to the compact expression

$$\rho_\hbar(\lambda) = \int dpf(p,\lambda) \tag{5.11}$$

for the eigenvalue density.

5.2 The mechanical analog

Since we are considering this ensemble to correspond to an equilibrium state, the natural thing is to impose the states $|\psi_m\rangle$ to be stationary states of the mechanical system, that is, eigenstates of a hamiltonian

$$\hat{H}|\psi_n\rangle = E_n|\psi_n\rangle\,. \tag{5.12}$$

Information about $\hat{H}$ can be derived from the representation of the wave function $\psi_N(x)$ as a matrix model correlator

$$\frac{1}{\sqrt{h_N}}e^{-\frac{V(x)}{2g_m}}\langle\det(x-M)\rangle = \psi_N(x)\,. \tag{5.13}$$

For g_m small (with 't Hooft coupling $t = g_m N$ fixed)

$$\langle \det(x-M) \rangle \approx e^{\langle \mathrm{tr} \log(x-M) \rangle} = e^{N \int^x dx R(x)} . \tag{5.14}$$

In this regime the resolvent is known to have the structure

$$R(x) \approx \frac{1}{2t} \left(V'(x) - y(x) \right) \tag{5.15}$$

from which we obtain

$$\frac{1}{\sqrt{h_N}} e^{-\frac{V(x)}{2g_m}} \langle \det(x-M) \rangle \approx \frac{1}{\sqrt{h_N}} e^{-\frac{1}{2g_m} \int^x y(x) dx} . \tag{5.16}$$

On the other hand, the first term in WKB approximation for $|\psi_N\rangle$ gives

$$\psi_N(x) \approx C e^{-\frac{i}{\hbar} \int^x p(x', E_N) dx'} . \tag{5.17}$$

Therefore

$$y(x) = ip(x, E_N) \qquad \int y(x) dx = iS(x, E_N) \qquad 2g_m = \hbar \tag{5.18}$$

which, after double-scaling limit, is the matrix model version of the map (3.5).

5.3 The double scaling limit

Let us briefly recall the definition of double scaling in one matrix models [6]. The relations between the orthogonal polynomials

$$\int d\lambda e^{-\frac{V}{g_m}} \Pi'_n \lambda \Pi_n = n \tag{5.19}$$

$$\int d\lambda e^{-\frac{V}{g_m}} \Pi'_n \lambda \Pi_n = \sqrt{r_n} \int d\lambda e^{-\frac{V}{g_m}} \frac{V'}{g_m} \Pi_n \Pi_{n-1} \tag{5.20}$$

with $r_n = h_n / h_{n-1}$, lead in the large $N \to \infty$ limit to the equation

$$t\xi = W(r(\xi)) + O\left(\frac{1}{N}\right) , \tag{5.21}$$

where $\xi = \frac{n}{N}$ and $r(\xi)$ the continuum function asoociated with r_n. The critical behavior of the $(2, 2k-1)$ minimal string is given by adjusting $V(\lambda)$ in such a way that, at some $r = r_c$, we have

$$W'(r = r_c) = W''(r = r_c) = \cdots = W^{k-1}(r = r_c) = 0 . \tag{5.22}$$

If we call $W(r = r_c) = t_c$, the double scaling limit is defined by

$$\frac{1}{N} = a^{2+\frac{1}{k}} \kappa \tag{5.23}$$

$$t_c - t\xi = a^2 \kappa^{-1/(1+\frac{1}{2k})} \tag{5.24}$$

$$\lambda = \lambda_c + a^{\frac{2}{k}} \tilde{\lambda} \tag{5.25}$$

$$a \to 0 , \tag{5.26}$$

where λ_c is the position of the edge of eingenvalue distribution. In this limit the parameter κ controls the genus expansion and becomes the string coupling constant g_s.

The effect of double scaling over the mechanical system associated with the matrix model is to transform the discrete set of $|\psi_n\rangle$ states into a continuum $|\psi_\xi\rangle$ in such a way that the matrix elements of $\hat{q}$ and $\hat{p}$ have continuum indices and become the Lax operator of the system. Due to the zoom (5.25) the mechanical system becomes an unbounded system with quantum corrections parametrized by κ. The eingenfunction ψ_{E_N}, now called $\psi_{E=0}$, which is known to give the all genus FZZT partition function,

$$\psi_{E=0}(x) = \frac{1}{\sqrt{h_N}} e^{-\frac{V(x)}{2g_m}} \langle \det(x-M)\rangle \Big|_{d.s.} \tag{5.27}$$

becomes exactly the Baker-Akhiezer function of the KP hierarchy [6].

6. The fold catastrophe and double scaling

We have seen in the last section that, from the point of view of the analog mechanical model, the double scaling is equivalent to perform a “zoom” near the turning point (notice that the turning point of the mechanical model corresponds to the border of the eigenvalue distribution). Thus, the strategy we will follow to derive the exact Baker-Akhiezer function is :

1. Starting with the matrix model, we define the corresponding bounded classical mechanical model.
2. Near the classical torus $p = p(q, E_N)$ we get a fold catastrophe for the Wigner function.
3. By uniform approximation we get a resolution of the fold catastrophe for the Wigner function near the classical torus.
4. We perform a “zoom” in q coordinate near the turning point.
5. We finally derive the Baker-Akhiezer function by integrating the Wigner function

$$\int dp f_{E=0}(p,q) = |\psi_{E=0}(q)|^2 . \tag{6.1}$$

Let us exemplify this procedure for the gaussian (2,1) matrix model. The large N resolvent is in this case

$$R(x) = \frac{1}{2t}\left(\lambda - \sqrt{\lambda^2 - 4t}\right). \tag{6.2}$$

According to the previous discussion the associated classical mechanical model is defined by

$$p(q, E = E_N) = \sqrt{E_N - q^2} \tag{6.3}$$

with $E_N = 4g_m N$. The torus in phase space, defined by $p = p(q, E)$, is a closed curve with the origin as its center of symmetry. From the resolution of the fold catastrophe on this curve (4.7) we have

$$\int_{-\infty}^{+\infty} dp f_{E_N}(p,q) = \frac{2\sqrt{2}}{\pi\hbar^{2/3}} \int_0^{+\infty} dp \frac{\left[\frac{3}{2}A(q,p)\right]^{1/6} Ai\left(-\left[\frac{3A(q,p)}{2\hbar}\right]^{2/3}\right)}{\left|\frac{\partial I}{\partial q}(2)\frac{\partial I}{\partial p}(1) - \frac{\partial I}{\partial q}(1)\frac{\partial I}{\partial p}(2)\right|^{1/2}} . \tag{6.4}$$

Once we perform the zoom into the turning point $q = +\sqrt{4g_m N}$ we obtain

$$dp = \frac{\left|\frac{\partial p}{\partial q}(q, E_N)\right|^{1/2} \left[\frac{3}{2}A(q,0)\right]^{1/6}}{2\sqrt{-\left[\frac{3}{2}A(q,p)\right]^{2/3} + \left[\frac{3}{2}A(q,0)\right]^{2/3}}} d\left[-\left[\frac{3}{2}A(q,p)\right]^{2/3}\right] \tag{6.5}$$

$$\left|\frac{\partial I}{\partial q}(2)\frac{\partial I}{\partial p}(1) - \frac{\partial I}{\partial q}(1)\frac{\partial I}{\partial p}(2)\right| \rightarrow 2\left|\frac{\partial p}{\partial q}(q, E_N)\frac{\partial I}{\partial p}\right| \tag{6.6}$$

with

$$p(q, E_N) \rightarrow p(q, E=0) = \sqrt{-\frac{q+1}{2}}\,. \tag{6.7}$$

From (6.1) we get:

$$|\psi_{E=0}|^2 = \frac{\left[\frac{3}{2}A(q,0)\right]^{1/3}}{\pi\hbar^{2/3}|\frac{\partial I}{\partial p}|} \int_{-\left[\frac{3}{2}A(q,0)\right]^{2/3}}^{+\infty} dV \frac{Ai\left(V/\hbar^{2/3}\right)}{\sqrt{V + \left[\frac{3}{2}A(q,0)\right]^{2/3}}} \tag{6.8}$$

Using now the magic "projection identity" [8]

$$\int_{-y}^{\infty} dx \frac{Ai(x)}{\sqrt{x+y}} = 2^{2/3}\pi Ai^2\left(\frac{y}{2^{2/3}}\right) \tag{6.9}$$

we get the well known result (4.3) [4] for the Baker-Akhiezer function in the double scaling limit of the gaussian model. This exercise is teaching us that the meaning of the double scaling is the quantum resolution of the fold catastrophe on the target curve at the classical turning point.

Note that this derivation of the Baker-Akhiezer function is general and independent of the type of critical point around which we do the double scaling limit. In the case of a critical point of order one ((2,1) model) the uniform approximation is simple and after the double scaling limit the function $\psi_{E=0}$ can be written in terms of the Airy function. On the other hand, in the case of $(2, 2k-1)$ models the resolution of the WKB singularity at the classical turning point is more complicated due to the fact that the classical turning point is of higher order. However, from the discussion in section 5.2 and 5.3, we conclude that the quantum resolution of the WKB singularity at the classical turning point encodes the same information as the double scaling limit and, therefore, as the Baker-Akhiezer function.

7. ZZ brane backgrounds and quantization rules

We have seen that, for $(2, 2k-1)$ models, the curve $p = p(q, E_N)$ near the turning point $q = q_t$ is of the form

$$p^2 \sim (q - q_t)^{2k-1}\,. \tag{7.1}$$

The singularity at the turning point is therefore a cusp for pure gravity (2,3) model and a ramphoid cusp for the (2,5) model. We can now consider the blow up resolution of these singularities. In the case of the cusp we only need to blow up once, which means one exceptional divisor. In the case of the ramphoid cusp the singularity can be resolved by blowing up twice. In general what we observe is that the resolution of the turning point singularity requires $k-1$ blow-ups. As it is well known, in this $(2,2k-1)$ models there are ZZ branes of type $(1,l)$ for $l=1,\ldots,k-1$ [2]. Thus the number of different types of ZZ branes is determined by the turning point singularity.

Let us consider backgrounds with ZZ branes. The deformation of the curve $M_{2,2k-1}$ due to a background with ZZ branes can be derived at first order from the FZZT-ZZ annulus amplitude [9]. This deformation is given by

$$\delta y^2 = -2^{2k-3}\sum_{l=1}^{k-1} g_s n_l \sqrt{x_{1,l}+1}\prod_{n\neq l}(x-x_{1,n})\,, \tag{7.2}$$

where n_l is the number of ZZ branes of the type $(1,l)$.

From now on we will consider only models with k odd that are well defined non perturbatively. The first thing to observe is that, for the matrix model effective potential, defined by

$$V_{\text{eff}}(x) = \text{Re}[+\int^x y dx] \tag{7.3}$$

backgrounds with n_l ZZ branes of type $(1,l)$ with l even correspond to have n_l eigenvalues at the stable saddle points i.e. the minima of V_{eff}, while backgrounds with ZZ branes of type $(1,l)$ with l odd correspond to have some eigenvalues at the maxima of V_{eff} [10]. This difference between ZZ backgrounds is reflected in the classical mechanical model in a very neat way. As we have already discussed, the classical curve on phase space for the $(2,2k-1)$ model contains a set of $k-1$ points on the q axis. These points are in correspondence with the different types of ZZ branes. Once we include the deformation of the curve due to the presence of ZZ branes what we get is:

(i) for backgrounds with ZZ branes of type $(1,l)$ with l even the corresponding point becomes a closed curve (see figure 5) in phase space,

(ii) for backgrounds with ZZ branes of type $(1,l)$ with l odd the corresponding point disappears. The reason for this phenomenon is quite simple. In fact the deformation of y^2 due to the presence of ZZ branes is to lift the saddles $(1,l)$ above or bellow the fixed energy line $y^2=0$ for l odd and even respectively (see figure 6).

7.1 Quantization Rules

Let us now consider the case of $(1,l)$ ZZ branes with l even. The corresponding curve $p=p(q,E)$ in phase space at the classical limit contains a closed curve Υ around the point $(1,l)$ (see figure 5). This automatically induces a quantization rule. A nice way to derive this quantization rule is imposing that the Wigner function should be single valued [8]. If we consider a closed path passing through the center of symmetry of Υ we obtain in saddle point approximation a phase factor for the Wigner function

$$e^{-\frac{i}{\hbar}\oint pdq+\pi i} \tag{7.4}$$

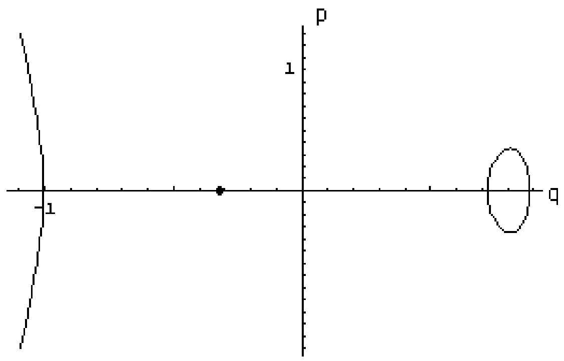

Figure 5: Deformed (2,5) model curve $p(q, E = 0)$ in a background with $ZZ_{1,2}$ branes.

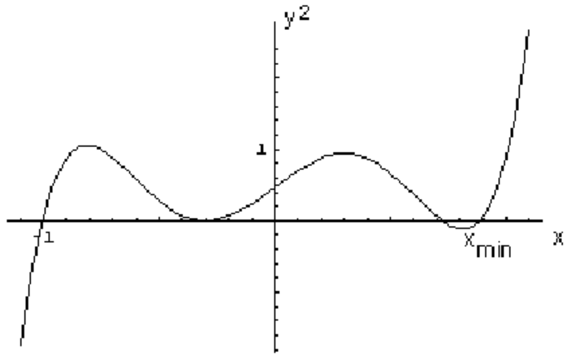

Figure 6: Mechanical system potential after $g_s n_2 = 0.01$ deformation.

which leads to the quantization condition

$$\frac{1}{2\pi} \oint pdq = \left(n + \frac{1}{2} \right) \hbar \,. \tag{7.5}$$

This quantization rule leads to the identification of the quantum number n with the number of $(1, l)$ ZZ branes with l even, and to the concrete identification between the Planck constant $\hbar$ of the mechanical model and the minimal string coupling constant g_s.

As an example, we can see these relations in more detail for the $(2, 5)$ model. In this case the deformed curve is given by (see figure 6)

$$y^2 = 8 \left[(x+1)(x - x_{1,1})^2 (x - x_{1,2})^2 - g_s n_2 \sqrt{x_{1,2} + 1}(x - x_{1,1}) \right] \tag{7.6}$$

with $x_{1,1} = \frac{1-\sqrt{5}}{4}$ and $x_{1,2} = \frac{1+\sqrt{5}}{4}$. If we consider small deformations $g_s n_2 \ll 1$ and expand around the (1,2) ZZ brane minimum we obtain the quadratic potential

$$y^2 \approx -\omega_1 g_s n_2 + \frac{\omega_2^2}{4} (x - x_{\text{min}})^2 \tag{7.7}$$

with $\omega_1 \simeq \omega_2 \simeq 12.3$. The level $E = 0$ satisfy the quantization condition (7.5) only if

$$\omega_1 g_s n_2 = \hbar \omega_2 n \tag{7.8}$$

from which we obtain that n_2 is a natural number and

$$\frac{\hbar}{g_s} = \frac{\omega_1}{\omega_2} \simeq 1\,. \tag{7.9}$$

8. Summary and open problems

In this paper we have suggested a general procedure to approach the quantum gravity effects on the target space of minimal strings. This procedure is based on the semiclassical mechanics in phase space as described by the Wigner function. The non perturbative effects encoded in the exact double scaling limit are in this scheme related to the type of singularity of the Wigner function on the classical target. As an example we have derived the double scaling limit of the gaussian model. For more general $(2, 2k-1)$ models we have described the type of singularity at the turning point. It remains to work out in full detail the resolution of the corresponding Wigner function in these cases. We have also derived a semiclassical quantization rule for ZZ branes, that obviously is not taking into account the tunneling unstabilities of these potentials. Concrete results on these tunneling rates requires the exact resolution of the Wigner function at the turning point for $(2, 2k-1)$ models.

From the point of view of topological strings on a Calabi-Yau defined by $uv+F(x,y) = 0$, a similar analysis can be done for the system associated to the curve $F(x,y) = 0$. The so defined Wigner function is very reminiscent of the Witten index for a BPS black hole [5]. We hope to address this fascinating issue in a future note.

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JHEP11(2005)049

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